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The Bounds of Vertex Padmakar–Ivan Index on k -Trees

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Abstract: The Padmakar–Ivan (PI) index is a distance-based topological index and a molecular structure descriptor, which is the sum of the number of vertices over all edges uv of a graph such that these vertices are not equidistant from u and v . In this paper, we explore the results of PI -indices from trees to recursively clustered trees, the k -trees. Exact sharp upper bounds of PI indices on k -trees are obtained by the recursive relationships, and the corresponding extremal graphs are given. In addition, we determine the PI -values on some classes of k -trees and compare them, and our results extend and enrich some known conclusions.

Keywords: extremal values; PI index; k -trees; distance

1. Introduction

Let G be a simple connected non-oriented graph with vertex set $V(G)$ and edge set $E(G)$. The distance $d(x, y)$ between the vertices $x, y \in V(G)$ is the minimum length of the paths between x and y in G . The oldest and most thoroughly examined molecular descriptor is Wiener index or path number [1], which was first considered in trees by Wiener in 1947 as follows: $W(G) = \sum_{\{x,y\} \subset V(G)} d(x, y)$. Compared to Wiener index, Szeged index was proposed by Gutman [2] in 1994 that, given $xy \in E(G)$, let $n_{xy}(x)$ be the number of vertices $w \in V(G)$ such that $d(x, w) < d(y, w)$, $Sz(G) = \sum_{xy \in E(G)} n_{xy}(x)n_{xy}(y)$. Based on the considerable success of Wiener index and Sz index, Khadikar proposed a new distance-based index [3] to be used in the field of nano-technology, that is edge Padmakar–Ivan (PI_e) index, $PI_e(G) = \sum_{xy \in E(G)} [n_e(x) + n_e(y)]$, where $n_e(x)$ denotes the number of edges which are closer to the vertex x than to the vertex y , and $n_e(y)$ denotes the number of edges which are closer to the vertex y than to the vertex x , respectively.

It is easy to see that the above concept does not count edges equidistant from both ends of the edge $e = xy$. Based on this idea, Khalifeh et al. [4] introduced a new PI index of vertex version that $PI(G) = PI_v(G) = \sum_{xy \in E(G)} [n_{xy}(x) + n_{xy}(y)]$. Note that, in order to obtain a good recursive formulas, we do not consider the vertices x, y for $n_{xy}(x)$ and $n_{xy}(y)$. Thus, $n_{xy}(x) + n_{xy}(y) \leq n - 2$.

Nowadays, Padmakar–Ivan indices are widely used in Quantitative Structure–Activity Relationship (QSAR) and Quantitative Structure–Property Relationship (QSPR) [5,6], and there are many interesting results [5,7–26] between graph theory and chemistry. For instances, Klavžar [27] provided PI -partitions and arbitrary Cartesian product. Pattabiraman and Paulraja [28] presented the formulas for vertex PI indices of the strong product of a graph and the complete multipartite graph. Ilić and Milosavljević [29] established basic properties of weighted vertex PI index and some lower and upper bounds on special graphs. Wang and Wei [30] studied vertex PI index on an extension of

trees (cacti). In [31], Das and Gutman obtained a lower bound on the vertex PI index of a connected graph in terms of numbers of vertices, edges, pendent vertices, and clique number. Hoji et al. [32] provided exact formulas for the vertex PI indices of Kronecker product of a connected graph G and a complete graph. Since the tree is a basic class of graphs in mathematics and chemistry, and these results indicate that either the stars or the paths attain the maximal or minimal bounds for particular chemical indices, then a natural question is how about the situations for vertex Padmakar–Ivan index?

Because PI index is a distance-based index and not very easy to calculate, we first consider the bipartite graph G with n vertices. Note that the tree is a subclass of bipartite graphs which have no odd cycles. By the definition of $PI(G)$ and the assumption that we do not consider the vertices x, y for $n_{xy}(x)$ and $n_{xy}(y)$, one can obtain that every edge of G has the PI -value as $n - 2$. Thus, the following observation is obtained.

Obervation 1. For a bipartite graph G with n vertices and m edges, $PI(G) = (n - 2)m$. In particular, if G is a tree, then $PI(G) = (n - 1)(n - 2)$.

Next, we will consider the graphs with odd cycles. In particular, the general tree, k -tree, contains a lot of odd cycles. Then, we are going to consider the PI indices of k -trees and figure out whether or not a k -star or a k -path attains the maximal or minimal bound for PI -indices of k -trees. Our main results are as follows: Theorems 1 and 2 give the exact PI -values of k -stars, k -paths and k -spirals (see Definitions 1–5 below).

Theorem 1. For any k -star S_n^k and k -path P_n^k with $n = kp + s$ vertices, where $p \geq 0$ is an integer and $s \in [2, k + 1]$, we have

$$\begin{aligned} (i) PI(S_n^k) &= k(n - k)(n - k - 1), \\ (ii) PI(P_n^k) &= \frac{k(k+1)(p-1)(3kp+6s-2k-4)}{6} + \frac{(s-1)s(3k-s+2)}{3}. \end{aligned}$$

Theorem 2. For any k -spiral $T_{n,c}^{k*}$ with $n \geq k$ vertices, where $c \in [1, k - 1]$, we have

$$PI(T_{n,c}^{k*}) = \begin{cases} \frac{(n-k)(n-k-1)(4k-n+2)}{3} & \text{if } n \in [k, 2k - c], \\ \frac{3c(n-2k+c-1)(n-2k+c) + (k-c)(2c^2+3nc-4kc+3kn-4k^2-6k+3n-2)}{3} & \text{if } n \geq 2k - c + 1. \end{cases}$$

Theorem 3 proves that k -stars achieve the maximal values of PI -values for k -trees, and Theorem 4 shows that k -paths do not arrive the minimal values and certain PI -values of k -spirals are less than that of k -paths.

Theorem 3. For any k -tree T_n^k with $n \geq k \geq 1$, we have $PI(T_n^k) \leq PI(S_n^k)$.

Theorem 4. For any k -spiral $T_{n,c}^{k*}$ with $n \geq k \geq 1$, we have

$$\begin{aligned} (i) \quad PI(P_n^k) &\geq PI(T_{n,c}^{k*}) \text{ if } c \in [1, \frac{k+1}{2}], \\ (ii) \quad PI(P_n^k) &\leq PI(T_{n,c}^{k*}) \text{ if } c \in [\frac{k+1}{2}, k - 1]. \end{aligned}$$

2. Preliminary

In this section, we first give some notations and lemmas that are crucial in the following sections. As usual, $G = (V, E)$ is a connected finite simple undirected graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $|G|$ or $|V|$ be the cardinality of V . For any $S \subseteq V(G)$ and $F \subseteq E(G)$, we use $G[S]$ to

denote the subgraph of G induced by S , $G - S$ to denote the subgraph induced by $V(G) - S$ and $G - F$ to denote the subgraph of G obtained by deleting F . $w(G - S)$ is the number of components of $G - S$ and S is a cut set if $w(G - S) \geq 2$. For any $u, v \in V(G)$, P_{uv} is a path connecting u and v , $d(u, v)$ is the distance between u and v , $N(v) = N_G(v) = \{w \in V(G), vw \in E(G)\}$ is the neighborhood of v and $N[v] = N(v) \cup \{v\}$. For any integers a, b with $a \leq b$, the interval $[a, b]$ is the set of all integers between a and b including a, b . In addition, let $[a, b) = [a, b] - \{b\}$ and $(a, b] = [a, b] - \{a\}$. In particular, $[a, b] = \emptyset$ for $a > b$. $f'(x)$ is a derivative of any differentiable function $f(x)$, where x is the variable. $\lfloor x \rfloor$ is the largest integer that is less than or equal to x ; $\lceil x \rceil$ is the smallest integer that is greater than or equal to x . It is clear that d is from 0 to the diameter of graphs. Other undefined notations are referred to [33].

It is commonly known that a chordal graph G with at least three vertices is a triangulated graph and contains a simplicial vertex, whose neighborhood induces a clique. During recent decades, there are many interesting studies related to chordal graphs. In 1969, Beineke and Pippert [7] gave the definition of k -trees, which is a significant subclass of chordal graphs. Now, we just give some definitions about k -trees below.

Definition 1. For positive integers n, k with $n \geq k$, the k -tree, denoted by T_n^k , is defined recursively as follows: The smallest k -tree is the k -clique K_k . If G is a k -tree with $n \geq k$ vertices and a new vertex v of degree k is added and joined to the vertices of a k -clique in G , then the obtained graph is a k -tree with $n + 1$ vertices.

Definition 2. For positive integers n, k with $n \geq k$, the k -path, denoted by P_n^k , is defined as follows: starting with a k -clique $G[\{v_1, v_2 \dots v_k\}]$. For $i \in [k + 1, n]$, the vertex v_i is adjacent to vertices $\{v_{i-1}, v_{i-2} \dots v_{i-k}\}$ only.

Definition 3. For positive integers n, k with $n \geq k$, the k -star, denoted by S_n^k , is defined as follows: Starting with a k -clique $G[\{v_1, v_2 \dots v_k\}]$ and an independent set S with $|S| = n - k$. For $i \in [k + 1, n]$, the vertex v_i is adjacent to vertices $\{v_1, v_2 \dots v_k\}$ only.

Definition 4. For positive integers n, k, c with $n \geq k$ and $c \in [1, k - 1]$, let v_1, v_2, \dots, v_{n-c} be the simplicial ordering of P_{n-c}^{k-c} . The k -spiral, denoted by $T_{n,c}^{k*}$, is defined as $P_{n-c}^{k-c} + K_c$, which is, $V(T_{n,c}^{k*}) = \{v_1, v_2, \dots, v_n\}$ and $E(T_{n,c}^{k*}) = E(P_{n-c}^{k-c}) \cup E(K_c) \cup \{v_1 v_l, v_2 v_l, \dots, v_{n-c} v_l\}$, for $l \in [n - c + 1, n]$.

Definition 5. Let $v \in V(T_n^k)$ be a vertex of degree k whose neighbors form a k -clique of T_n^k , then v is called a k -simplicial vertex. Let $S_1(T_n^k)$ be the set of all k -simplicial vertices of T_n^k , for $n \geq k + 2$, and set $S_1(K_k) = \emptyset, S_1(K_{k+1}) = \{v\}$, where v is any vertex of K_{k+1} . Let $G_0 = G, G_i = G_{i-1} - v_i$, where v_i is a k -simplicial vertex of G_{i-1} , then $\{v_1, v_2 \dots v_n\}$ is called a simplicial elimination ordering of the n -vertex graph G .

In order to consider the PI -value of any k -tree G , let $G' = G \cup \{u\}$ be a k -tree obtained by adding a new vertex u to G . For any $v_1, v_2 \in V(G)$, let $d(v_1, v_2)$ be the distance between v_1 and v_2 in G , $d'(v_1, v_2)$ be the distance between v_1 and v_2 in G' . Now, we define a function that measures the difference of PI -values of any edge relating a vertex from G to G' as follows: $f : \{w \in V(G'), xy \in E(G)\}$ to $\{1, 0\}$ as follows:

$$f(w, xy) = \begin{cases} 0, & \text{if } w = u \text{ and } d'(x, w) = d'(y, w), \\ 0, & \text{if } w \in V(G) \text{ and } d(x, w) - d'(x, w) = d(y, w) - d'(y, w), \\ 1, & \text{if otherwise.} \end{cases}$$

Using the construction of k -trees, we can derive the following lemmas. Note that $PI(xy) = n_{xy}(x) + n_{xy}(y)$ and $PI(xy) \leq n - 2$.

Lemma 1. Let xy be any edge of a k -tree G with at least $n \geq k + 1$ vertices, then $PI(xy) \leq n - k - 1$.

Proof. Since every vertex of any k -tree G with at least $k + 1$ vertices must be in some $(k + 1)$ -cliques, which is, $|N(x) \cap N(y)| \geq k - 1$ for any $xy \in E(G)$, we have $PI(xy) \leq n - (k - 1) - 2 = n - k - 1$. \square

Lemma 2. Let xy be any edge of a k -tree G with n vertices and $G' = G \cup \{u\}$ be a k -tree obtained by adding u to G . If $w \in V(G)$, then $f(w, xy) = 0$.

Proof. By adding u to G , since G' is a k -tree, we can get that the distance of any pair of vertices of G will increase at most 1, then $f(w, xy) \leq 1$. If $w \in V(G)$, then there exists a shortest path P_{xw} or P_{yw} such that $u \notin V(P_{xw})$ or $V(P_{yw})$, that is, $f(w, xy) = 0$. \square

Lemma 3. For any k -path G with n vertices, where $n \geq k + 2$, let $S_1(G) = \{v_1, v_n\}$ and $\{v_1, v_2, \dots, v_n\}$ be the simplicial elimination ordering of G , then $d(v_i, v_j) = \lceil \frac{j-i}{k} \rceil$, for $i < j$ and $i, j \in [1, n]$. Furthermore, if $n = kp + s$ with $p \geq 1, s \in [2, k + 1]$, then

$$d(v, v_{kp+s}) = \begin{cases} p + 1 & \text{if } v \in \{v_1, v_2, \dots, v_{s-1}\}, \\ p - i & \text{if } v \in \{v_{ki+s}, v_{ki+s+1}, \dots, v_{k(i+1)+s-1}\}, i \in [0, p - 1]. \end{cases}$$

Proof. If $j - i \leq k$, then v_i, v_j must be in the same $(k + 1)$ -clique of G , and we have $d(v_i, v_j) = 1$; if $j - i \geq k + 1$, then $P_{v_i v_j} = v_i v_{i+k} v_{i+2k} \dots v_{i+(\lfloor \frac{j-i}{k} \rfloor - 1)k} v_{i+\lfloor \frac{j-i}{k} \rfloor k} v_j$ is one of the shortest paths between v_i and v_j . Thus, $d(v_i, v_j) = \lceil \frac{j-i}{k} \rceil$ and Lemma 3 is proved. \square

Lemma 4. For any k -spiral $T_{n,c}^{k*}$ with n vertices and $v_i, v_j \in V(T_{n,c}^{k*})$ for $i < j$,

$$d(v_i, v_j) = \begin{cases} 1, & \text{if } j - i \leq k - c, i, j \in [1, n - c], \\ 1, & \text{if } i \text{ or } j \in [n - c + 1, n], \\ 2, & \text{if } j - i \geq k - c + 1, i, j \in [1, n - c]. \end{cases}$$

Proof. If $j - i \leq k - c$ with $i, j \in [1, n - c]$, by Definition 4, we can get that v_i, v_j must be in the same $(k + 1)$ -clique of G and $d(v_i, v_j) = 1$; If i or $j \in [n - c + 1, n]$, without loss of generality, say v_i such that $i \in [n - c + 1, n]$, then $N[v_i] = V(T_{n,c}^{k*})$, that is, $d(v_i, v_j) = 1$; If $j - i \geq k - c + 1$ with $i, j \in [1, n - c]$, then $v_i \notin N(v_j)$ and $P_{v_i v_j} = v_i v_n v_j$ is one of the shortest paths between v_i and v_j , that is, $d(v_i, v_j) = 2$. Thus, Lemma 4 is proved. \square

3. Main Proofs

In this section, we give the proofs of main results by inductions. For a k -tree T_n^k , if $n = k$ or $k + 1$, then T_n^k is a k or $(k + 1)$ -clique, that is, $PI(T_n^k) = 0$. Thus, all of the theorems are true and we will only consider the case when $n \geq k + 2$ below.

Proof of Theorem 1. For (i), let $V(S_n^k) = \{u_1, u_2, \dots, u_n\}$, $G[\{u_1, \dots, u_k\}]$ be a k -clique and $N(u_{l_0}) = \{u_1, u_2, \dots, u_k\}$ for $l_0 \geq k + 1$. Just by Definition 3, we can get that for $i, j \in [1, k]$, $N[u_i] = N[u_j] = V(S_n^k)$, then $PI(u_i u_j) = n_{u_i u_j}(u_i) + n_{u_i u_j}(u_j) = 0$; for $i \in [1, k]$ and $l_0 \in [k + 1, n]$, $|N[u_i] - N[u_{l_0}]| = n - k - 1$, then $PI(u_i u_{l_0}) = n - k - 1$. Thus, we can get $PI(S_n^k) = \sum_{i,j \in [1,k]} PI(u_i u_j) + \sum_{i \in [1,k], l_0 \in [k+1,n]} PI(u_i u_{l_0}) = k(n - k)(n - k - 1)$.

For (ii), we will proceed it by induction on $|P_n^k| = n \geq k + 2$. If $n = k + 2$, let $\{v_1, v_2, \dots, v_{k+2}\}$ be the simplicial elimination ordering of P_{k+2}^k . By Lemma 3, we can get that $PI(v_1 v_i) = 1, PI(v_i v_{i'}) = 0$ and $PI(v_i v_{k+2}) = 1$ for $i, i' \in [2, k + 1]$. Thus, $PI(P_{k+2}^k) = \sum_{i=2}^{k+1} PI(v_1 v_i) + \sum_{i=2}^{k+1} PI(v_i v_{k+2}) = 2k$. Assume that Theorem 1 is true for a k -path with at most $kp + s - 1$ vertices, where $p \geq 1, 2 \leq s \leq k + 1$. Let P_n^k be a k -path such that $|P_n^k| = kp + s, V(P_n^k) = \{v_1, v_2, \dots, v_{kp+s}\}$ and $\{v_1, v_2, \dots, v_{kp+s}\}$ be

the simplicial elimination ordering of P_n^k . Set $P_{n-1}^k = P_n^k - \{v_{kp+s}\}$, then $\{v_1, v_2, \dots, v_{kp+s-1}\}$ is the simplicial elimination ordering of P_{n-1}^k and for any edge $v_i v_j \in E(P_n^k)$, $d(v_i, v_j)$ or $d'(v_i, v_j)$ is the distance of v_i and v_j in P_{n-1}^k or P_n^k , respectively.

Let $\alpha = \left[\frac{k(k+1)(p-1)(3kp+6s-2k-4)}{6} + \frac{(s-1)s(3k-s+2)}{3} \right] - \left[\frac{k(k+1)(p-1)(3kp+6s-2k-10)}{6} + \frac{(s-2)(s-1)(3k-s+3)}{3} \right] = pk^2 + pk - k^2 - 3k + 2ks - s^2 + 3s - 2$. If we can show that by adding v_{kp+s} to P_{n-1}^k , $PI(P_n^k) = PI(P_{n-1}^k) + \alpha$, then Theorem 1 is true.

Set $w = v_{kp+s}$, $A_1 = \{v_1 v_s, v_1 v_{s+1}, \dots, v_1 v_{k+1}\}$, $A_2 = \{v_2 v_s, \dots, v_2 v_{k+2}\}, \dots, A_{s-1} = \{v_{s-1} v_s, \dots, v_{s-1} v_{k+s-1}\}$ and $B_1 = \{v_1 v_2, v_1 v_3, \dots, v_1 v_{s-1}\}$, $B_2 = \{v_2 v_3, \dots, v_2 v_{s-1}\}, \dots, B_{s-2} = \{v_{s-2} v_{s-1}\}$, $B_{s-1} = \emptyset$. By Definition 2 and Lemma 3, we have $d'(v_1, v_{kp+s}) = p+1$, $d'(v_s, v_{kp+s}) = p$ and $d'(v_1, v_{kp+s}) = p+1$, $d'(v_2, v_{kp+s}) = p+1$, that is, $d'(v_1, v_{kp+s}) \neq d'(v_s, v_{kp+s})$ and $d'(v_1, v_{kp+s}) = d'(v_2, v_{kp+s})$. Thus, $f(w, v_1 v_s) = 1$ and $f(w, v_1 v_2) = 0$. Similarly, for any edge $v_{h_1} v_{h_2} \in \cup_{i=1}^{s-1} A_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v_{kp+s}) \neq d'(v_{h_2}, v_{kp+s})$, that is, $f(w, v_{h_1} v_{h_2}) = 1$; For $v_{h_1} v_{h_2} \in \cup_{i=1}^{s-1} B_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v_{kp+s}) = d'(v_{h_2}, v_{kp+s})$, that is, $f(w, v_{h_1} v_{h_2}) = 0$. Thus, we can get that

$$f(v_{kp+s}, xy) = \begin{cases} 1, & \text{if } xy \in \cup_{i=1}^{s-1} A_i, \\ 0, & \text{if } xy \in \cup_{i=1}^{s-1} B_i. \end{cases}$$

For $t \in [0, p-2]$, set $A_{kt+s} = \{v_{kt+s} v_{k(t+1)+s}\}$, $A_{kt+s+1} = \{v_{kt+s+1} v_{k(t+1)+s}, v_{kt+s+1} v_{k(t+1)+s+1}\}, \dots, A_{k(t+1)+s-1} = \{v_{k(t+1)+s-1} v_{k(t+1)+s}, v_{k(t+1)+s-1} v_{k(t+1)+s+1}, \dots, v_{k(t+1)+s-1} v_{k(t+2)+s-1}\}$, and $B_{kt+s} = \{v_{kt+s} v_{kt+s+1}, \dots, v_{kt+s} v_{k(t+1)+s-1}\}$, $B_{kt+s+1} = \{v_{kt+s+1} v_{kt+s+2}, \dots, v_{kt+s+1} v_{k(t+1)+s-1}\}, \dots, B_{k(t+1)+s-2} = \{v_{k(t+1)+s-2} v_{k(t+1)+s-1}\}$, $B_{k(t+1)+s-1} = \emptyset$. For $t = 0$ and by Lemma 3, we have $d'(v_s, v_{kp+s}) = p$, $d'(v_{k+s}, v_{kp+s}) = p-1$ and $d'(v_s, v_{kp+s}) = p$, $d'(v_{s+1}, v_{kp+s}) = p$, that is, $d'(v_s, v_{kp+s}) \neq d'(v_{k+s}, v_{kp+s})$ and $d'(v_s, v_{kp+s}) = d'(v_{s+1}, v_{kp+s})$. Thus, $f(w, v_s v_{k+s}) = 1$ and $f(w, v_s v_{s+1}) = 0$. similarly, for any edge $v_{h_1} v_{h_2} \in \cup_{i=kt+s}^{k(t+1)+s-1} A_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v_{kp+s}) \neq d'(v_{h_2}, v_{kp+s})$, that is, $f(w, v_{h_1} v_{h_2}) = 1$; for $v_{h_1} v_{h_2} \in \cup_{i=kt+s}^{k(t+1)+s-1} B_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v_{kp+s}) = d'(v_{h_2}, v_{kp+s})$, that is, $f(w, v_{h_1} v_{h_2}) = 0$. Thus, we can get that

$$f(v_{kp+s}, xy) = \begin{cases} 1, & \text{if } xy \in \cup_{i=kt+s}^{k(t+1)+s-1} A_i, \\ 0, & \text{if } xy \in \cup_{i=kt+s}^{k(t+1)+s-1} B_i. \end{cases}$$

Next, we consider the edges in the $(k+1)$ -clique $P_n^k[N[v_{kp+s}]]$. For any edge $v_{h_1} v_{h_2}$ with $h_1, h_2 \in [k(p-1)+s, kp+s-1]$, we have $d'(v_{h_1}, v_{kp+s}) = d'(v_{h_2}, v_{kp+s}) = 1$, that is, $f(w, v_{h_1} v_{h_2}) = 0$. For any edge $v_h v_{kp+s}$ with $h \in [k(p-1)+s, kp]$, by Lemma 3, we can obtain that $d'(v_1, v_h) = p$, $d'(v_1, v_{kp+s}) = p+1$, $d'(v_{h-k}, v_h) = 1$, $d'(v_{h-k}, v_{kp+s}) = 2$ and when $h \neq k(p-1)+s$, $d'(v_{k(p-1)+s}, v_h) = 1$, $d'(v_{k(p-1)+s}, v_{kp+s}) = 1$, that is, $d'(v_1, v_h) \neq d'(v_1, v_{kp+s})$, $d'(v_{h-k}, v_h) \neq d'(v_{h-k}, v_{kp+s})$ and $d'(v_{k(p-1)+s}, v_h) = d'(v_{k(p-1)+s}, v_{kp+s})$. Similarly, we get that for $j \in [1, p-1]$, $j' \in [1, p]$ and $l \neq h$,

$$\begin{cases} d'(v_l, v_h) \neq d'(v_l, v_{kp+s}) & \text{if } l \in [1, s-1] \cup [h-jk, k(p-j)+s-1], \\ d'(v_l, v_h) = d'(v_l, v_{kp+s}) & \text{if } l \in [k(p-j')+s, h-j'k+k-1] \cup [h+1, kp+s-1]. \end{cases}$$

Thus, if $v_h = v_{k(p-1)+s}$, then $d'(v_l, v_{k(p-1)+s}) \neq d'(v_l, v_{kp+s})$ with $l \in [1, s-1] \cup \{\cup_{j=1}^{p-1} [k(p-1)+s-jk, (p-j)k+s-1]\} = [1, (p-1)k+s-1]$ and $d'(v_l, v_{k(p-1)+s}) = d'(v_l, v_{kp+s})$ with $l \in [(p-1)k+s+1, kp+s]$, that is, $PI(v_{k(p-1)+s} v_{kp+s}) = (p-1)k+s-1$; similarly, we can obtain that $PI(v_{k(p-1)+s+1} v_{kp+s}) = (p-1)(k-1)+s-1$; $PI(v_{k(p-1)+s+2} v_{kp+s}) = (p-1)(k-2)+s-1$; \dots ; $PI(v_{kp} v_{kp+s}) = (p-1)s+s-1$.

For any edge $v_h v_{kp+s}$ with $h \in [kp+1, kp+s-1]$, by Lemma 3, we can obtain that $d'(v_{h-k}, v_h) = 1$, $d'(v_{h-k}, v_{kp+s}) = 2$ and $d'(v_{k(p-1)+s}, v_h) = 1$, $d'(v_{k(p-1)+s}, v_{kp+s}) = 1$, that is, $d'(v_{h-k}, v_h) \neq$

$d'(v_{h-k}, v_{kp+s})$ and $d'(v_{k(p-1)+s}, v_h) = d'(v_{k(p-1)+s}, v_{kp+s})$. Similarly, we get that for $j'' \in [1, p]$ and $l \neq h$,

$$\begin{cases} d'(v_l, v_h) \neq d'(v_l, v_{kp+s}) & \text{if } l \in [h - j''k, k(p - j'') + s - 1], \\ d'(v_l, v_h) = d'(v_l, v_{kp+s}) & \text{if } l \in [k(p - j'') + s, h - j''k + k - 1] \cup [h + 1, kp + s - 1]. \end{cases}$$

Thus, if $v_h = v_{kp+1}$, then $d'(v_l, v_{kp+1}) \neq d'(v_l, v_{kp+s})$ for $l \in \cup_{j''=1}^p [kp + 1 - j''k, k(p - j'') + s - 1]$ and $d'(v_l, v_{kp+1}) = d'(v_l, v_{kp+s})$ with $l \in \{\cup_{j''=1}^p [k(p - j'') + s, k(p + 1 - j'')]\} \cup [h + 1, kp + s - 1]$, that is, $PI(v_{kp+1}v_{kp+s}) = (s - 1)p$; similarly, we have $PI(v_{kp+1}v_{kp+s}) = (s - 2)p; \dots; PI(v_{kp+s-2}v_{kp+s}) = 2p; PI(v_{kp+s-1}v_{kp+s}) = p$.

Set $w \in V(P_n^k)$, if $xy \in E(P_n^k)$ with x or $y \neq v_{kp+s}$, by Lemma 2, we have $f(w, xy) = 0$. Thus,

$$\begin{aligned} PI(P_n^k) - PI(P_{n-1}^k) &= \sum_{xy \in \cup_{i=1}^{k(p-1)+s-1} (A_i \cup B_i)} f(w, xy) + PI(v_{k(p-1)+s}v_{kp+s}) \\ &\quad + PI(v_{k(p-1)+s+1}v_{kp+s}) + \dots + PI(v_{kp+s-1}v_{kp+s}) \\ &= [(k + 2 - s) + (k + 3 - s) + \dots + k] + (1 + 2 + \dots + k)(p - 1) \\ &\quad + [k(p - 1) + s - 1] + [(k - 1)(p - 1) + s - 1] + [(k - 2)(p - 1) + s - 1] \\ &\quad + \dots + [s(p - 1) + s - 1] + [(s - 1)p + (s - 2)p + \dots + 2p + p] \\ &= pk^2 + pk - k^2 - 3k + 2ks - s^2 + 3s - 2 \\ &= \alpha. \end{aligned}$$

Thus, $PI(P_n^k) = \frac{k(k+1)(p-1)(3kp+6s-2k-4)}{6} + \frac{(s-1)s(3k-s+2)}{3}$, for $|P_n^k| = kp + s$ and Theorem 1 is proved. \square

Proof of Theorem 2. We will proceed with it by induction on $n \geq k + 2$. If $n = k + 2$, by Definition 4, we have $T_{n,c}^{k*}$ is also a k -path, that is, $PI(T_{n,c}^{k*}) = 2k$. If $n \geq k + 3$, assume that Theorem 2 is true for the k -spiral with at most $n - 1$ vertices, we will consider $T_{n,c}^{k*}$ with n vertices. Let $T_{n,c}^{k*}$ be a k -spiral with $V(T_{n,c}^{k*}) = V(T_{n-1,c}^{k*}) \cup \{v\}$ and $E(T_{n,c}^{k*}) = E(T_{n-1,c}^{k*}) \cup \{vv_{n-1}, vv_{n-2}, \dots, vv_{n-k}\}$ such that $v_1, v_2, \dots, v_{n-c-1}$ is the simplicial ordering of P_{n-c-1}^{k-c} , where $T_{n-1,c}^{k*} = P_{n-c-1}^{k-c} + K_c$ with $V(T_{n-1,c}^{k*}) = \{v_1, v_2, \dots, v_{n-1}\}$ and $E(T_{n-1,c}^{k*}) = E(P_{n-c-1}^{k-c}) \cup E(K_c) \cup \{v_1v_l, v_2v_l, \dots, v_{n-c-1}v_l\}$ for $l \in [n - c, n - 1]$. For any edge $v_i v_j \in E(T_{n,c}^{k*})$, $d(v_i, v_j)$ or $d'(v_i, v_j)$ is the distance of v_i and v_j in $T_{n-1,c}^{k*}$ or $T_{n,c}^{k*}$, respectively.

For $k + 2 \leq n \leq 2k - c$, let $\gamma = \frac{(n-k)(n-k-1)(4k-n+2)}{3} - \frac{(n-k-1)(n-k-2)(4k-n+3)}{3} = (n - k - 1)(3k - n + 2)$. If we can show that by adding v to $T_{n-1,c}^{k*}$, $PI(T_{n,c}^{k*}) = PI(T_{n-1,c}^{k*}) + \gamma$, then Theorem 2 is true.

Set $w = v$ and let $l \in [n - c, n - 1]$, by Lemma 4, we have $d'(v_l, v) = 1$ and $d'(v_i, v) = 2$ for $i \in [1, n - k - 1]$, that is, $f(w, v_l v_i) = 1$; $d'(v_l, v) = d'(v_i, v) = 1$ for $i \in [n - k, n - 1]$, that is, $f(w, v_l v_i) = 0$. Set $C_1 = \{v_1 v_2, v_1 v_3, \dots, v_1 v_{n-k-1}\}$, $C_2 = \{v_2 v_3, v_2 v_4, \dots, v_2 v_{n-k-1}\}, \dots, C_{n-k-2} = \{v_{n-k-2} v_{n-k-1}\}$, $C_{n-k-1} = \emptyset$, $D_1 = \{v_1 v_{n-k}, v_1 v_{n-k+1}, \dots, v_1 v_{k-c+1}\}$, $D_2 = \{v_2 v_{n-k}, v_2 v_{n-k+1}, \dots, v_2 v_{k-c+2}\}, \dots, D_{n-k-1} = \{v_{n-k-1} v_{n-k}, v_{n-k-1} v_{n-k+1}, \dots, v_{n-k-1} v_{n-c-1}\}$. By Lemma 4, we have $d'(v_1, v) = d'(v_2, v) = 2$ and $d'(v_{n-k}, v) = 1$, that is, $f(w, v_1 v_2) = 0$ and $f(w, v_1 v_{n-k}) = 1$. Similarly, for $v_{h_1} v_{h_2} \in \cup_{i=1}^{n-k-1} C_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v) = d'(v_{h_2}, v) = 2$, that is, $f(w, v_{h_1} v_{h_2}) = 0$; for $v_{h_1} v_{h_2} \in \cup_{i=1}^{n-k-1} D_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v) = 2$ and $d'(v_{h_2}, v) = 1$, that is, $f(w, v_{h_1} v_{h_2}) = 1$. Set $C_{n-k} = \{v_{n-k} v_{n-k+1}, v_{n-k} v_{n-k+2}, \dots, v_{n-k} v_{n-c-1}\}$, $C_{n-k+1} = \{v_{n-k+1} v_{n-k+2}, v_{n-k+1} v_{n-k+3}, \dots, v_{n-k+1} v_{n-c-1}\}, \dots, C_{n-c-2} = \{v_{n-c-2} v_{n-c-1}\}$. By Lemma 4, we have $d'(v_{n-k}, v) = d'(v_{n-k-1}, v) = 1$, that is, $f(w, v_{n-k} v_{n-k-1}) = 0$. Similarly, for $v_{h_1} v_{h_2} \in \cup_{i=n-k}^{n-c-2} C_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v) = d'(v_{h_2}, v) = 1$, that is, $f(w, v_{h_1} v_{h_2}) = 0$.

Set $E_1 = \{v v_i, i \in [n - k, n - c - 1]\}$, by Lemma 4, we have $d'(v_i, v) = 2, d'(v_i, v_{n-k}) = 1$ for $i \in [1, n - k - 1]$ and $d'(v_j, v) = d'(v_j, v_{n-k}) = 1$ for $i \in [n - k + 1, n]$. Thus, $PI(v_{n-k} v) = n - k - 1$. Similarly, $PI(v_{n-k+1} v) = PI(v_{n-k+2} v) = \dots = PI(v_{k-c+1} v) = n - k - 1$. In addition, by Lemma 4, we have $d'(v_i, v) = 2, d'(v_i, v_{k-c+2}) = 1$ for $i \in [2, n - k - 1]$, $d'(v_1, v) = d'(v_1, v_{k-c+2}) = 2$ and $d'(v_j, v) = d'(v_j, v_{k-c+2}) = 1$ for $j \in [n - k, n]$. Thus, $PI(v_{k-c+2} v) = n - k - 2$. Similarly, we have $PI(v_{k-c+3} v) = n - k - 3, PI(v_{k-c+4} v) = n - k - 4, \dots, PI(v_{n-c-1} v) = 1$. Set $E_2 = \{v v_l, l \in$

$[n - c, n - 1]$, since $N[v_l] - N[v] = n - k - 1$, we have $PI(vv_l) = n - k - 1$. Set $E_3 = \{v_i v_l, i \in [1, n - c - 1], l \in [n - c, n - 1]\}$, by Lemma 4, we have $d'(v_i, v) = 2$ for $i \in [1, n - k - 1]$, $d'(v_i, v) = 1$ for $i \in [n - k, n - c - 1]$, $d'(v_l, v) = 1$ for $l \in [n - c, n - 1]$. Thus, $f(w, v_i v_l) = 1$ for $i \in [1, n - k - 1]$ and $f(w, v_i v_l) = 0$ for $i \in [n - k, n - c - 1]$.

Set $w \in V(T_n^{k*}) - \{v\}$, if $xy \in E(T_n^{k*})$ with x or $y \neq v$, by Lemma 2, we have $f(w, xy) = 0$. Thus,

$$\begin{aligned} PI(T_n^{k*}) - PI(T_{n-1}^{k*}) &= \sum_{xy \in \cup_{i=1}^{n-c-2} C_i} f(w, xy) + \sum_{xy \in \cup_{i=1}^{n-k-1} D_i} f(w, xy) + \sum_{xy \in E_1 \cup E_2} PI(xy) + \\ &\quad \sum_{xy \in E_3} f(w, xy) \\ &= 0 + [(2k - n - c + 2) + (2k - n - c + 3) + \cdots + (k - c)] \\ &\quad + [1 + 2 + \cdots + (n - k - 2) + (n - k - 1)(2k - n - c + 2)] \\ &\quad + c(n - k - 1) + c(n - k - 1) \\ &= (n - k - 1)(3k - n + 2) \\ &= \gamma, \end{aligned}$$

and Theorem 2 is proved.

For $n \geq 2k - c + 1$, let $\sigma = \frac{3c(n-2k+c-1)(n-2k+c)+(k-c)(2c^2+3nc-4kc+3kn-4k^2-6k+3n-2)}{3} - \frac{3c(n-2k+c-2)(n-1-2k+c)+(k-c)(2c^2+3(n-1)c-4kc+3k(n-1)-4k^2-6k+3n-2)}{3} = k^2 - 4kc + c^2 + 2nc - 3c + k$. If we can show that by adding v to $T_{n-1,c}^{k*}$, $PI(T_n^{k*}) = PI(T_{n-1,c}^{k*}) + \sigma$, then Theorem 2 is proved.

Set $w = v$, by Lemma 4, we have $d'(v_l, v) = 1$ for $l \in [n - c, n - 1]$, $d'(v_i, v) = 2$ for $i \in [1, n - k - 1]$ and $d'(v_j, v) = 1$ for $j \in [n - k, n - c - 1]$. Thus, $f(w, v_l v_i) = 1$ and $f(w, v_l v_j) = 0$. Set $C_1 = \{v_1 v_2, v_1 v_3, \dots, v_1 v_{k-c+1}\}$, $C_2 = \{v_2 v_3, v_2 v_4, \dots, v_2 v_{k-c+2}\}, \dots, C_{n-2k+c-1} = \{v_{n-2k+c-1} v_{n-2k+c}, v_{n-2k+c-1} v_{n-2k+c+1}, \dots, v_{n-2k+c-1} v_{n-k-1}\}$, $C_{n-2k+c} = \{v_{n-2k+c} v_{n-2k+c+1}, v_{n-2k+c} v_{n-2k+c+2}, \dots, v_{n-2k+c} v_{n-k-1}\}$, $C_{n-2k+c+1} = \{v_{n-2k+c+1} v_{n-2k+c+2}, v_{n-2k+c+1} v_{n-2k+c+3}, \dots, v_{n-2k+c+1} v_{n-k-1}\}, \dots, C_{n-k-1} = \phi$, $D_{n-2k+c} = \{v_{n-2k+c} v_{n-k}\}$, $D_{n-2k+c+1} = \{v_{n-2k+c+1} v_{n-k}, v_{n-2k+c+1} v_{n-k+1}\}, \dots, D_{n-k-1} = \{v_{n-k-1} v_{n-k}, v_{n-k-1} v_{n-k+1}, \dots, v_{n-k-1} v_{n-c-1}\}$.

By Lemma 4, we can get that $d'(v_1, v) = d'(v_2, v) = 2$ and $d'(v_{n-k}, v) = 1$, that is, $f(w, v_1 v_2) = 0$ and $f(w, v_1 v_{n-k}) = 1$. Similarly, for $v_{h_1} v_{h_2} \in \cup_{i=1}^{n-k-1} C_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v) = d'(v_{h_2}, v) = 2$, that is, $f(w, v_{h_1} v_{h_2}) = 0$; for $v_{h_1} v_{h_2} \in \cup_{i=n-2k+c}^{n-k-1} D_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v) = 2$ and $d'(v_{h_2}, v) = 1$, that is, $f(w, v_{h_1} v_{h_2}) = 1$. Set $C_{n-k} = \{v_{n-k} v_{n-k+1}, v_{n-k} v_{n-k+2}, \dots, v_{n-k} v_{n-c-1}\}$, $C_{n-k+1} = \{v_{n-k+1} v_{n-k+2}, v_{n-k+1} v_{n-k+3}, \dots, v_{n-k+1} v_{n-c-1}\}, \dots, C_{n-c+2} = \{v_{n-c-2} v_{n-c-1}\}$. By Lemma 4, we can get that $d'(v_{n-k}, v) = d'(v_{n-k+1}, v) = 1$, that is, $f(w, v_{n-k} v_{n-k+1}) = 0$. Similarly, for $v_{h_1} v_{h_2} \in \cup_{i=n-k}^{n-c-2} C_i$ with $h_1 < h_2$, we have $d'(v_{h_1}, v) = d'(v_{h_2}, v) = 1$, that is, $f(w, v_{h_1} v_{h_2}) = 0$.

Set $E_1 = \{v v_i, i \in [n - k, n - c - 1]\}$, by Lemma 4, we have $d'(v, v_{n-k-1}) = 2, d'(v_{n-c-1}, v_{n-k-1}) = 1, d'(v, v_i) = d'(v_{n-c-1}, v_i) = 1$ for $i \in [n - k, n - c - 2] \cup [n - c, n - 1]$ and $d'(v, v_j) = d(v_{n-c-1}, v_j) = 2$ for $j \in [1, n - k - 2]$. Thus, $PI(vv_{n-c-1}) = 1$. Similarly, we have $PI(vv_{n-c-2}) = 2, PI(vv_{n-c-3}) = 3, \dots, PI(vv_{n-k}) = k - c$. Set $E_2 = \{v v_l, l \in [n - c, n - 1]\}$, since $N[v_l] - N[v] = n - k - 1$, we have $PI(vv_l) = n - k - 1$. Set $E_3 = \{v_i v_l, i \in [1, n - c - 1], l \in [n - c, n - 1]\}$, by Lemma 4, we have $d'(v, v_i) = 2, d'(v, v_l) = 1$ for $i \in [1, n - k - 1]$ and $d'(v, v_i) = d'(v, v_l) = 1$ for $i \in [n - k, n - c - 1]$. Thus, $f(w, v_i v_l) = 1$ for $i \in [1, n - k - 1]$ and $f(w, v_i v_l) = 0$ for $i \in [n - k, n - c - 1]$.

Set $w \in V(T_n^{k*}) - \{v\}$, if $xy \in E(T_n^{k*})$ with x or $y \neq v$, by Lemma 2, we have $f(w, xy) = 0$. Thus,

$$\begin{aligned} PI(T_n^{k*}) - PI(T_{n-1}^{k*}) &= \sum_{xy \in \cup_{i=1}^{n-c-2} C_i} f(w, xy) + \sum_{xy \in \cup_{i=n-2k+c}^{n-k-1} D_i} f(w, xy) + \sum_{xy \in E_1 \cup E_2} PI(xy) + \\ &\quad + \sum_{xy \in E_3} f(w, xy) \\ &= 0 + [1 + 2 + 3 + \cdots + (k - c)] + [1 + 2 + 3 + \cdots + (k - c)] \\ &\quad + c(n - k - 1) + c(n - k - 1) \\ &= k^2 - 4kc + c^2 + 2nc - 3c + k \\ &= \sigma, \end{aligned}$$

and Theorem 2 is proved. \square

Proof of Theorem 3. For $n \geq k + 2$, we will proceed it by induction on $|T_n^k| = n$. If $n = k + 2$, T_n^k is also a k -path, that is, $PI(T_n^k) = 2k$. If $n \geq k + 3$, assume that Theorem 3 is true for the k -tree with at most $n - 1$ vertices, let $v \in S_1(T_n^k)$ and $T_{n-1}^k = T_n^k - v$, by the induction hypothesis, we have $PI(T_{n-1}^k) \leq PI(S_{n-1}^k) = k(n - k - 1)(n - k - 2)$. By adding back v , let $N(v) = \{x_1, x_2, \dots, x_k\}$ and $w = v$. Since $T_n^k[v, x_1, x_2, \dots, x_k]$ is a $(k + 1)$ -clique, we have $f(w, x_i x_j) = 0$ for $i, j \in [1, k]$. By Lemmas 1 and 2, we can obtain that $PI(vx_i) \leq n - k - 1$ with $i \in [1, k]$ and $f(w, xy) \leq 1$ for any edge $xy \in E(T_n^k) - E(T_n^k[v, x_1, x_2, \dots, x_k])$. Next, set $w \in V(T_n^k) - \{v\}$, by Lemma 2, if $xy \in E(T_n^k)$ with x or $y \neq v$, we have $f(w, xy) = 0$. Since $|E(T_n^k) - E(T_n^k[v, x_1, x_2, \dots, x_k])| = k(n - k - 1)$, we have

$$\begin{aligned} PI(T_n^k) &= PI(T_{n-1}^k) + \sum_{xy \in E(T_n^k) - \{vx_i, i \in [1, k]\}} f(w, xy) + \sum_{i=1}^k PI(vx_i) \\ &\leq PI(S_{n-k}^k) + k(n - k - 1) + k(n - k - 1) \\ &= k(n - k - 1)(n - k - 2) + k(n - k - 1) + k(n - k - 1) \\ &= k(n - k)(n - k - 1) \\ &= PI(S_n^k). \end{aligned}$$

Thus, this finishes the proof of Theorem 3. \square

Proof of Theorem 4. For $k = 1$ and by Fact 1, we can get that every tree with the same vertices has the same PI -value; then, Theorem 4 is obvious; for $k \geq 2$, if $k + 2 \leq n \leq 2k - c$, let $n = kp + s$ with $p = 1$ and $s = n - k$, by Theorem 1, we have $PI(P_n^k) - PI(T_{n,c}^{k*}) = \frac{(s-1)s(3k-s+2)}{3} - \frac{(n-k)(n-k-1)(4k-n+2)}{3} = \frac{(n-k-1)(n-k)[3k-(n-k)+2]}{3} - \frac{(n-k)(n-k-1)(4k-n+2)}{3} = 0$, and Theorem 4 is true. If $n \geq 2k - c + 1$, $p = \frac{n-s}{k}$ and by Theorems 1 and 2, define the new functions as follows: for $z \geq 2k - c + 1$, $1 \leq c \leq k - 1$ and $2 \leq s \leq k + 1$,

$$\begin{aligned} g(z) &= \frac{(k+1)(z-s-k)(3z+3s-2k-4)}{6} + \frac{s(s-1)(3k-s+2)}{3}, \\ h(z, c) &= c(z - 2k + c - 1)(z - 2k + c) + \frac{(k-c)(2c^2+3zc-4kc+3kz-4k^2-6k+3z-2)}{3}, \\ l(z, c) &= g(z) - h(z, c) \\ &= \left(\frac{k}{2} + \frac{1}{2} - c\right)z^2 + (-c^2 + 2c + 4kc - \frac{11k^2}{6} - \frac{5k}{2} - \frac{2}{3})z \\ &\quad + \frac{ks^2}{2} - \frac{k^2s}{6} - \frac{ks}{2} + \frac{5k^3}{3} + \frac{5k^2}{3} + \frac{s^2}{2} + \frac{4k}{3} - \frac{s^3}{3} - 6k^2c + 3kc^2 - \frac{c^3}{3} - 4kc + c^2 - \frac{2c}{3}, \\ l_z(z) &= l_z(z, c) \\ &= (k + 1 - 2c)z - c^2 + 2c + 4kc - \frac{11k^2}{6} - \frac{5k}{2} - \frac{2}{3}. \end{aligned}$$

Then, it is enough to determine whether or not $l(z, c) \geq 0$ is true. By some calculations, we can obtain the following claim:

Claim 1. $z_1 = 2k - c + 1$, $z_2 = 2k - c + 2$ are the two roots of $l(z, c) = 0$ with $c \neq \frac{k+1}{2}$.

Proof. For any $c \in [1, k - 1]$, let $z_1 = 2k - c + 1$, $z_2 = 2k - c + 2$, and we have $l(2k - c + 1, c) = 0$, $l(2k - c + 2, c) = 0$. If $c \neq \frac{k+1}{2}$, then Claim is true. \square

For fixed $c \in [1, \frac{k+1}{2})$, that is, $\frac{k}{2} + \frac{1}{2} - c > 0$, then the function of $l(z, c)$ about z is open up. Since z is an integer and by Fact 2, we have $l(z, c) \geq 0$ for $z \geq 2k - c + 1$ and Theorem 4 is true; if $c = \frac{k+1}{2}$ and $k \geq 1$, we have $l_z(z) = \frac{1-k^2}{12} \leq 0$, that is, $l(z, \frac{k+1}{2})$ is decreasing about z . By the proof of Fact 2, we have $l(2k - c + 1, c) = 0$. For $z \geq 2k - c + 1$, we can get that $l(z, \frac{k+1}{2}) \leq l(2k - c + 1, \frac{k+1}{2}) = 0$ and Theorem 4 is true; for fixed $c \in (\frac{k+1}{2}, k - 1]$, that is, $\frac{k}{2} + \frac{1}{2} - c < 0$, then the function of $l(z, c)$ about z is open down. Since z is an integer and by Claim, we can obtain that $l(z, c) \leq 0$ for $z \geq 2k - c + 1$ and this finishes the proof of Theorem 4. \square

4. Conclusions

We can see that the k -stars attain the maximal values of PI -values for k -trees. One of the guesses is that the k -paths attain the minimal values. Actually, it is not the case and some PI -values of k -spirals

is even smaller than that of k -paths. Meanwhile, not all PI -values of k -spirals are less than the values of all other k -trees. This fact indicates an interesting problem—which type of k -trees will achieve the minimum PI -value?

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